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Numerical Solution of a Hyperbolic-Parabolic Problem with Nonlocal Boundary Conditions

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ABSTRACT

A numerical method is proposed for solving hyperbolic-parabolic partial differential equations with nonlocal boundary condition. The first and second orders of accuracy difference schemes are presented. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of a one-dimensional hyperbolic-parabolic partial differential equations. The method is illustrated by numerical examples.

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1. INTRODUCTION

Methods of solutions of nonlocal boundary value problems for hyperbolic-parabolic differential equations have been studied extensively by many researchers. (Vallet, 2003; Glazatov, 1998; Karatoprakliev, 1989; Gerish, Kotschote and Zacher, 2004; Vragov, 1983; Nakhushev, 1995; Ramos, 2006; Liu, Cui and Sun, 2006; Berdyshev and Karimov, 2006; Salakhitdinov and Urinov, 1997; Dzhuraev, 1978; Bazarov and Soltanov, 1995; Ashyralyev and Yurtsever, 2005; Ashyralyev and Orazov, 1999).

In (Ashyralyev and Ozdemir, 2007), the nonlocal boundary value problem for differential equations

$$\begin{cases} \dfrac{d^2u(t)}{dt^2}+Au(t)=f(t)(0\le t\le 1), \dfrac{du(t)}{dt}Au(t)=g(t)(-1\le t\le 0), \\ u(-1)=\displaystyle\sum_{j=1}^{K}\alpha_j u(\mu_j)+\sum_{j=1}^{L}\beta_j\frac{du(\lambda_j)}{dt}+\varphi, \sum_{j=1}^{K}\left|\alpha_j\right|, \sum_{j=1}^{L}\left|\beta_j\right|\le 1, 0<\mu_j,\lambda_j\le 1 \end{cases} \tag{1}$$

in a Hilbert space H with self-adjoint positive definite operator A was considered.

The stability estimates for the solution of problem (1) were established. In applications, the stability estimates for the solution of mixed type boundary value problems for hyperbolic-parabolic equations were obtained.

In the present paper, the application results of (Ashyralyev and Ozdemir, 2007) to numerical solutions of difference schemes of nonlocal boundary value problems for the multi-dimensional hyperbolic-parabolic equation are considered. The stability estimates for the solution of difference schemes of the nonlocal boundary value problem for the multi-dimensional hyperbolic-parabolic equation are obtained. A procedure of modified Gauss elimination method is used for solving these difference schemes in the case of a one-dimensional hyperbolic-parabolic partial differential equation. The method is illustrated by numerical examples.

2. DIFFERENCE SCHEMES AND STABILITY ESTIMATES

Let Ω be the unit open cube in the n-dimensional Euclidean space $\mathbb{R}^n \left(0 < x_k < 1,\ 1 \le k \le n\right)$ with boundary $S, \bar{\Omega} = \Omega \cup S$. In $[0,1] \times \Omega$, the mixed boundary value problem for the multi-dimensional hyperbolic-parabolic equation

$$\begin{cases} u_{tt} - \sum_{r=1}^{n} \left(a_r(x) u_{x_r}\right)_{x_r} = f(t,x), 0 \le t \le 1, x = (x_1, \dots, x_n) \in \Omega, \\ u_t - \sum_{r=1}^{n} \left(a_r(x) u_{x_r}\right)_{x_r} = g(t,x), -1 \le t \le 0, x = (x_1, \dots, x_n) \in \Omega, \\ u(-1,x) = \sum_{j=1}^{K} \alpha_j u(\mu_j, x) + \sum_{j=1}^{L} \beta_j u_t(\lambda_j, x) + \varphi(x), x \in \tilde{\Omega}, \\ \sum_{j=1}^{K} |\alpha_j|, \sum_{j=1}^{L} |\beta_j| \le 1, 0 < \mu_j, \lambda_j \le 1 \\ u(t,x) = 0, x \in S, -1 \le t \le 1 \end{cases} \tag{2}$$

is considered, where $a_r(x),(x\in\Omega),\varphi(x)\left(x\in\overline{\Omega}\right),f(t,x)\left(t\in[0,1],x\in\Omega\right)$, $g(t,x)\left(t\in[-1,0],x\in\Omega\right)$ are smooth functions and $a_r(x)\geq a>0$. The discretization of problem (2) is carried out in two steps. In the first step, let us define the grid sets

$$\tilde{\Omega}_h=\{x=x_m=(h_1m_1,...,h_nm_n),m=(m_1,...,m_n),$$
$$0\leq m_r\leq N_r,h_rN_r=1,r=1,...,n\},$$
$$\Omega_h=\tilde{\Omega}_h\cap\Omega,S_h=\tilde{\Omega}_h\cap S.$$

We introduce the Hilbert space $L_{2h}=L_2\left(\tilde{\Omega}_h\right)$ of grid functions $\varphi^h(x)=\{\varphi(h_1m_1,...,h_nm_n)\}$ defined on $\tilde{\Omega}_h$, equipped with the norm

$$\left\|\varphi^h\right\|_{L_2\left(\tilde{\Omega}_h\right)}=\left(\sum_{x\in\overline{\Omega}_h}\left|\varphi^h(x)\right|^2h_1\cdots h_n\right)^{1/2}.$$

To the differential operator A generated by problem (2), we assign the difference operator A_h^x by the formula

$$A_h^xu_x^h=-\sum_{r=1}^n\left(a_r(x)u_{\bar{x}_r}^h\right)_{x_r,j_r} \tag{3}$$

acting in the space of grid functions $u^h(x)$, satisfying conditions $u^h(x)=0$ for all $x\in S_h$. It is known that A_h^x is a self-adjoint positive definite operator in L_{2h}. With the help of A_h^x, we arrive at the nonlocal boundary value problem

$$
\begin{cases}
\dfrac{d^{u}u^{h}(t,x)}{dt^{2}}+A_{h}^{x}u^{h}(t,x)=f^{h}(t,x),0\le t\le 1,x\in\Omega_{h},\\
\dfrac{du^{h}(t,x)}{dt}+A_{h}^{x}u^{h}(t,x)=g^{h}(t,x),-1\le t\le 0,x\in\Omega_{h},\\
u^{h}(-1,x)=\displaystyle\sum_{j=1}^{K}\alpha_{j}u^{h}(\mu_{j},x)+\sum_{j=1}^{L}\beta_{j}\dfrac{du^{h}(\lambda_{j},x)}{dt}+\varphi^{h}(x),x\in\tilde{\Omega}_{h},\\
\displaystyle\sum_{j=1}^{K}\left|\alpha_{j}\right|,\sum_{j=1}^{L}\left|\beta_{j}\right|\le 1,0<\mu_{j},\lambda_{j}\le 1,\\
u^{h}(0^{+},x)=u^{h}(0^{-},x),\dfrac{du^{h}(0^{+},x)}{dt}=\dfrac{du^{h}(0^{-},x)}{dt},x\in\tilde{\Omega}_{h}
\end{cases}
\qquad (4)
$$

for an infinite system of ordinary differential equations.

In the second step, problem (4) is replaced by difference schemes in paper (Ashyralyev and Ozdemir, 2005). So, we have

$$
\begin{cases}
\tau^{-2}\left(u_{k+1}^{h}(x)-2u_{k}^{h}(x)+u_{k-1}^{h}(x)\right)+A_{h}^{x}u_{k+1}^{h}(x)=f_{k}^{h}(x),\\
f_{k}^{h}(x)=f^{h}(t_{k+1},x_{n}),t_{k+1}=(k+1)\tau,1\le k\le N-1,N\tau=1,x\in\Omega_{h},\\
\tau^{-1}\left(u_{k}^{h}(x)-u_{k-1}^{h}(x)\right)+A_{h}^{x}u_{k}^{h}(x)=g_{k}^{h}(x),\\
g_{k}^{h}(x)=g^{h}(t_{k},x_{n}),t_{k}=k\tau,-N+1\le k\le -1,x\in\Omega_{h},\\
u_{-N}^{h}(x)=\displaystyle\sum_{j=1}^{K}\alpha_{j}u_{[\mu_{j}/\tau]}^{h}(x)+\sum_{j=1}^{L}\beta_{j}\tau^{-1}\left(u_{[\lambda/\tau]}^{h}(x)-u_{[\lambda/\tau]-1}^{h}(x)\right)\\
+\varphi^{h}(x),x\in\tilde{\Omega}_{h},\\
\tau^{-1}\left(u_{1}^{h}(x)-u_{0}^{h}(x)\right)=-A_{h}^{x}u_{0}^{h}(x)+g_{0}^{h}(x)=g^{h}(0,x),x\in\tilde{\Omega}_{h},
\end{cases}
\qquad (5)
$$

and two types of second order of accuracy difference schemes

$$\begin{cases}
\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+A_h^x u_k^h(x)+\frac{\tau^2}{4}\left(A_h^x\right)^2 u_{k+1}^h(x)=f_k^h(x), \\
f_k^h(x)=f^h(t_k,x), t_k=k\tau, 1\le k\le N-1, x\in\Omega_h, \\
\tau^{-1}(I+\tau^2 A_h^x)(u_1^h(x)-u_0^h(x))=Z_1, \\
Z_1=\frac{\tau}{2}(f^h(0,x)-A_h^x u_0^h(x))+(g^h(0,x)-A_h^x u_0^h(x)), x\in\tilde{\Omega}_h, \\
\tau^{-1}\left(u_k^h(x)-u_{k-1}^h(x)\right)+A_h^x(I+\frac{\tau}{2}A_h^x)u_k^h(x)=(I+\frac{\tau}{2}A_h^x)g_k^h(x), \\
g_k^h(x)=g^h(t_k-\frac{\tau}{2},x), t_k=k\tau, -(N-1)\le k\le 0, x\in\Omega_h, \\
u_{-N}^h(x)=\sum_{j=1}^{j}\alpha_j\left(u_{[\mu_j/\tau]}^h(x)+\left(\mu_j-[\mu_j/\tau]\tau\right)\tau^{-1}\left(u_{[\mu_j/\tau]}^h(x)-u_{[\mu_j/\tau]-1}^h(x)\right)\right) \\
+\sum_{j=1}^{L}\beta_j\left(\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)\right)+\left(\lambda_j-[\lambda_j/\tau]\tau+\frac{\tau}{2}\right) \\
\times\left(f_{[\lambda_j/\tau]}^h(x)-A_h^x u_{[\lambda_j/\tau]}^h(x)\right)+\varphi^h(x), 2\tau<\mu_j, 2\tau<\lambda_j, x\in\tilde{\Omega}_h
\end{cases} \tag{6}$$

$$\begin{cases}\tau^{-2}\left(u_{k+1}^h(x)-2u_k^h(x)+u_{k-1}^h(x)\right)+\frac{1}{2}A_h^x u_k^h(x)\\ +\frac{1}{4}A_h^x\left(u_{k+1}^h(x)+u_{k-1}^h(x)\right)=f_k^h(x),\\ f_k^h(x)=f^h(t_k,x),t_k=k\tau,1\le k\le N-1,x\in\Omega_h,\\ \tau^{-1}\left(I+\tau^2A_h^x\right)\left(u_1^h(x)-u_0^h(x)\right)=Z_1,\\ Z_1=\frac{\tau}{2}\left(f^h(0,x)-A_h^x u_0^h(x)\right)+\left(g^h(0,x)-A_h^x u_0^h(x)\right),x\in\tilde{\Omega}_h,\\ \tau^{-1}\left(u_k^h(x)-u_{k-1}^h(x)\right)+A_h^x\left(I+\frac{\tau}{2}A_h^x\right)u_k^h(x)=\left(I+\frac{\tau}{2}A_h^x\right)g_k^h(x),\\ g_k^h(x)=g^h\left(t_k-\frac{\tau}{2},x\right),t_k=k\tau,-(N-1)\le k\le 0,x\in\Omega_h,\\ u_{-N}^h(x)=\sum_{j=1}^{K}\alpha_j\left(u_{[\mu_j/\tau]}^h(x)+\left(\mu_j-[\mu_j/\tau]\tau\right)\tau^{-1}\left(u_{[\mu_j/\tau]}^h(x)-u_{[\mu_j/\tau]-1}^h(x)\right)\right)\\ +\sum_{j=1}^{L}\beta_j\left(\tau^{-1}\left(u_{[\lambda_j/\tau]}^h(x)-u_{[\lambda_j/\tau]-1}^h(x)\right)\right)+\left(\lambda_j-[\lambda_j/\tau]\tau+\frac{\tau}{2}\right)\\ \times\left(f_{[\lambda_j/\tau]}^h(x)-A_h^x u_{[\lambda_j/\tau]}^h(x)\right)+\varphi^h(x),2\tau<\mu_j,2\tau<\lambda_j,x\in\tilde{\Omega}_h\end{cases}\tag{7}$$

are obtained.

Theorem 1. Let τ and $|h|$ be sufficiently small numbers. Then, the solution of difference scheme (5) satisfies the following stability estimates:

$$\begin{aligned}&\max_{-N\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{-N+1\le k\le N}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^{n}\left\|\left(u_k^h\right)_{x_r,j_r}\right\|_{L_{2h}}\\ &\le M_1\left[\left\|f_1^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\left(f_k^h-f_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\left\|g_0^h\right\|_{L_{2h}}\right.\\ &\left.+\max_{-N+1\le k\le 0}\left\|\left(g_k^h-g_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(\varphi^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}\right],\end{aligned}$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$+\max_{-N\le k\le N}\sum_{r=1}^{n}\left\|\left(u_k^h\right)_{\bar{x}_r x_r,j_r}\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$\le M_1\left[\sum_{r=1}^{n}\left\|\left(f_1^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(f_2^h-f_1^h\right)\right\|_{L_{2h}}\right.$$

$$+\max_{2\le k\le N-1}\left\|\tau^{-2}\left(f_{k+1}^h-2f_k^h+f_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(g_0^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(g_0^h-g_{-1}^h\right)\right\|_{L_{2h}}$$

$$\left.+\max_{-N+1\le k\le -1}\left\|\tau^{-2}\left(g_{k+1}^h-2g_k^h+g_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(\varphi^h\right)_{\bar{x}_r x_r,j_r}\right\|_{L_{2h}}\right].$$

Here, M_1 does not depend on $\tau, h, \varphi^h(x)$ and $f_k^h(x), 1\le k<N, g_k^h(x)$, $-N<k\le 0$.

Theorem 2. Let τ and $|h|$ be sufficiently small numbers. Then, for the solutions of difference schemes (6) and (7) the following stability inequalities

$$\max_{-N\le k\le N}\left\|u_k^h\right\|_{L_{2h}}+\max_{-N+1\le k\le N}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}+\max_{-N\le k\le N}\sum_{r=1}^{n}\left\|\left(u_k^h\right)_{x_r,j_r}\right\|_{L_{2h}}$$

$$\le M_2\left[\left\|f_0^h\right\|_{L_{2h}}+\max_{2\le k\le N-1}\left\|\left(f_k^h-f_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\left\|g_0^h\right\|_{L_{2h}}\right.$$

$$\left.+\max_{-N+1\le k\le 0}\left\|\left(g_k^h-g_{k-1}^h\right)\tau^{-1}\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(\varphi^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}\right],$$

$$\max_{1\le k\le N-1}\left\|\tau^{-2}\left(u_{k+1}^h-2u_k^h+u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$+\max_{-N\le k\le N}\sum_{r=1}^{n}\left\|\left(u_k^h\right)_{\bar{x}_r x_r,j_r}\right\|_{L_{2h}}+\max_{-N+1\le k\le 0}\left\|\tau^{-1}\left(u_k^h-u_{k-1}^h\right)\right\|_{L_{2h}}$$

$$\le M_2\left[\sum_{r=1}^{n}\left\|\left(f_0^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(f_1^h-f_0^h\right)\right\|_{L_{2h}}\right.$$

$$+\max_{2\le k\le N-1}\left\|\tau^{-2}\left(f_{k+1}^h-2f_k^h+f_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(g_0^h\right)_{\bar{x}_r,j_r}\right\|_{L_{2h}}+\left\|\tau^{-1}\left(g_0^h-g_{-1}^h\right)\right\|_{L_{2h}}$$

$$+\max_{-N+1\le k\le -1}\left\|\tau^{-2}\left(g_{k+1}^h-2g_k^h+g_{k-1}^h\right)\right\|_{L_{2h}}+\sum_{r=1}^{n}\left\|\left(\varphi^h\right)_{\bar{x}_r x_r, j_r}\right\|_{L_{2h}}\Bigg].$$

hold, where M_2 is independent of not only $\tau, h, \varphi^h(x)$ but also $f_k^h(x), 1\le k<N, g_k^h(x), -N<k\le 0.$

Proofs of Theorems 1-2 are based on symmetry properties of the operator A_h^x defined by formula (3) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in L_{2h}.

Theorem 3. For the solution of the elliptic difference problem

$$A_h^x u^h(x)=\omega^h(x), x\in\Omega_h,$$
$$u^h(x)=0, x\in S_h$$

the following coercivity inequality holds (Sobolevskii, 1975)

$$\sum_{r=1}^{n}\left\|\left(u^h\right)_{\bar{x}_r x_r, j_r}\right\|_{L_{2h}}\le M_3\left\|\omega^h\right\|_{L_{2h}}.$$

3. NUMERICAL RESULTS

We have not been able to obtain a sharp estimate for constants figuring in the stability inequalities. Therefore, the following result of numerical experiments of the nonlocal boundary value problem

$$\begin{cases} u_{tt}-u_{xx}=\left(-2+\pi^2+4t^2\right)e^{-t^2}\sin\pi x, 0<t<1, 0<x<1,\\ u_t-u_{xx}=\left(-2+\pi^2\right)e^{-t^2}\sin\pi x, -1<t<0, 0<x<1,\\ u\left(0^+,x\right)=u\left(0^-,x\right), u_t\left(0^+,x\right)=u_t\left(0^-,x\right), 0\le x\le 1,\\ u(-1,x)=\frac{1}{4}u\left(\frac{1}{2},x\right)+\frac{1}{4}u_t\left(\frac{1}{2},x\right)+\frac{1}{4}u(1,x)+\frac{1}{4}u_t(1,x)\\ +\frac{5}{4e}\sin\pi x, 0\le x\le 1,\\ u(t,0)=u(t,1)=0, -1\le t\le 1, \end{cases} \tag{8}$$

for hyperbolic-parabolic equation is considered.

First, applying first order of accuracy difference scheme (5), we get system of equations in matrix form

$$AU_{n+1} + BU_n + CU_{n-1} = D\varphi_n, 1 \le n \le M-1;\ U_0 = U_M = \tilde{0},$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 \\ 0 & a & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 \\ 0 & 0 & a & \cdot & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a & \cdot & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 & \cdot & a & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 & a \\ 0 & 0 & 0 & \cdot & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & k & l & \cdot & 0 & k & l \\ b & c & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & b & c & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & d & e & f & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & d & e & f \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 1 & -2 & 1 & \cdot & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$C = A$ and D is $(2N+1)\times(2N+1)$ identity matrix

$$U_s = \begin{bmatrix} U_s^{-N} \\ \cdots \\ U_s^{0} \\ \cdots \\ U_s^{N} \end{bmatrix}_{(2N+1)\times 1}, \varphi_n = \begin{bmatrix} \varphi_n^{-N} \\ \cdots \\ \varphi_n^{0} \\ \cdots \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1} \quad \text{for } s = n \pm 1, n.$$

Also here

$$\begin{cases} a = -\dfrac{1}{h^2}, b = \dfrac{1}{\tau}, c = \dfrac{1}{\tau} + \dfrac{2}{h^2}, \\ d = \dfrac{1}{\tau^2}, e = -\dfrac{2}{\tau^2}, f = \dfrac{1}{\tau^2} + \dfrac{2}{h^2}, \\ k = \dfrac{1}{4\tau} l = -\dfrac{1}{4} - \dfrac{1}{4\tau}, \end{cases} \quad \text{and } \varphi_n^k = \begin{cases} \dfrac{5}{4e}\sin(\pi x_n), k = -N, \\ g(t_k, x_n), -N+1 \le k \le 0, \\ f(t_{k+1}, x_n), 1 \le k \le N-1, \\ 0, k = N. \end{cases}$$

So, we have the second order difference equation with respect to n with matrix coefficients. To solve this difference equation, we have applied a procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we seek a solution of the matrix equation in the following form

$$U_j = \alpha_{j+1} U_{j+1} + \beta_{j+1}, j = M-1, ..., 2, 1, U_M = \tilde{0},$$

where $\alpha_j \left(j = 1, ..., M-1 \right)$ are $\left(2N+1\right) \times \left(2N+1\right)$ square and $\beta_j \left(j = 1, ..., M-1 \right)$ are $\left(2N+1\right) \times 1$ column matrices defined by formulas

$$\begin{cases} \alpha_{j+1} = -\left(B + C\alpha_j\right)^{-1} A, \\ \beta_{j+1} = \left(B + C\alpha_j\right)^{-1}\left(D\varphi_j - C\beta_j\right), j = 1, ..., M-1. \end{cases}$$

Here, α_1 and β_1 are both zero matrices whose dimensions are $\left(2N+1\right) \times \left(2N+1\right)$ and $\left(2N+1\right) \times 1$, respectively.

Second, applying second order difference scheme (6) and simply formulas

$$\frac{2u(0)-5u(h)+4u(2h)-u(3h)}{h^2}-u''(0)=O\left(h^2\right),$$
$$\frac{2u(1)-5u(1-h)+4u(1-2h)-u(1-3h)}{h^2}-u''(1)=O\left(h^2\right),$$
$$\frac{u\left(x_{n+2}\right)-4u\left(x_{n+1}\right)+6u\left(x_n\right)-4u\left(x_{n-1}\right)+u\left(x_{n-2}\right)}{h^4}-u^{lv}\left(x_n\right)=O\left(h^2\right),$$

we get system of equations in matrix form

$$\begin{cases} AU_{n+2}+BU_{n+1}+CU_n+DU_{n-1}+EU_{n-2}=R\varphi_n, 2\leq n\leq M-2, \\ U_0=\tilde{0}, U_M=\tilde{0}, U_1=\frac{4}{5}U_2-\frac{1}{5}U_3, U_{M-2}-\frac{1}{5}U_{M-3}, \end{cases} \tag{9}$$

where

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & w & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & w & d & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & w & d \\ 0 & 0 & 0 & \cdot & m & n & \cdot & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & r & s & t & \cdot & r & s & t \\ -\frac{1}{\tau} & e & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & -\frac{1}{\tau} & e & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & \frac{1}{\tau^2} & f & g & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & \frac{1}{\tau^2} & f & g \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & p & q & \cdot & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$D=B, E=A$ and R is $(2N+1)\times(2N+1)$ identity matrix and

$$U_s=\begin{bmatrix} U_s^{-N} \\ \cdots \\ U_s^{0} \\ \cdots \\ U_s^{N} \end{bmatrix}_{(2N+1)\times 1}, \quad \varphi_n=\begin{bmatrix} \varphi_n^{-N} \\ \cdots \\ \varphi_n^{0} \\ \cdots \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1} \quad \text{for } s=n\pm 2, n\pm 1, n.$$

Also here,

$$\begin{cases} a=\dfrac{\tau}{2h^4}, b=\dfrac{\tau^2}{4h^4}, c=\dfrac{1}{h^2}-\dfrac{2\tau}{h^4}, d=-\dfrac{\tau^2}{h^4}, w=-\dfrac{1}{h^2}, m=\dfrac{\tau-2}{2h^2}, \\ n=-\dfrac{\tau}{h^2}, e=\dfrac{1}{\tau}+\dfrac{2}{h^2}+\dfrac{3\tau}{h^4}, f=-\dfrac{2}{\tau^2}+\dfrac{2}{h^2}, g=\dfrac{1}{\tau^2}+\dfrac{3\tau^2}{2h^4}, \\ p=-\dfrac{1}{\tau}+\dfrac{3\tau+2}{h^2}, q=\dfrac{1}{\tau}+\dfrac{2\tau}{h^2}, r=-\dfrac{1}{8\tau}, s=\dfrac{1}{2\tau}, t=-\dfrac{1}{4}-\dfrac{3}{8\tau}, \end{cases}$$

and

$$\varphi_n^k=\begin{cases} \dfrac{5}{4e}\sin\left(\pi x_n\right), k=-N, \\ g\left(t_k-\dfrac{\tau}{2}, x_n\right)-\dfrac{\tau}{2h^2}\left[g\left(t_k-\dfrac{\tau}{2}, x_{n+1}\right)-2g\left(t_k-\dfrac{\tau}{2}, x_n\right)+g\left(t_k-\dfrac{\tau}{2}, x_{n-1}\right)\right] \\ -\dfrac{\tau}{4h}\left[g\left(t_k-\dfrac{\tau}{2}, x_{n+1}\right)-g\left(t_k-\dfrac{\tau}{2}, x_{n-1}\right)\right], -N+1\le k\le 0, \\ f\left(t_k, x_n\right), 1\le k\le N-1, \\ 0, k=N. \end{cases}$$

So, we have the fourth order difference equation with respect to n with matrix coefficients. To solve this difference equation, we have applied another procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients, namely

$$U_j=\alpha_{j+1}U_{j+1}+\beta_{j+1}U_{j+2}+\gamma_{j+1}, j=M-2,...,1,0, \tag{10}$$

where $\alpha_j, \beta_j\left(j=1,...,M-1\right)$ are $\left(2N+1\right)\times\left(2N+1\right)$ square and $\gamma_j, \left(j=1,...,M-1\right)$ are $\left(2N+1\right)\times 1$ column matrices defined by formulas

$$\begin{cases} \alpha_{j+1}=-(C_n+D\alpha_j+E\beta_{j-1}+E\alpha_{j-1}\alpha_j)^{-1}(B+D\beta_j+E\alpha_{j-1}\beta_j), \\ \beta_{j+1}=-(C+D\alpha_j+E\beta_{j-1}+E\alpha_{j-1}\alpha_j)^{-1}A, \\ \gamma_{j+1}=(C_n+D\alpha_j+E\beta_{j-1}+E\alpha_{j-1}\alpha_j)^{-1}(R\varphi_j-D\gamma_j-E\alpha_{j-1}\gamma_j-E\gamma_{j-1}), \end{cases} \tag{11}$$

where $j=2,...,M-2.$ Here, α_1 and β_1 are $(2N+1)\times(2N+1)$ zero matrices, I is $(2N+1)\times(2N+1)$ identity matrix, $\alpha_2=\frac{4}{5}I,$ $\beta_2=\frac{1}{5}I,\gamma_1$ and γ_2 are $(2N+1)\times 1$ zero matrices and

$$\begin{cases} U_M=\tilde{0}; U_{M-1}=P\left[(4I-\alpha_{M-2})\gamma_{M-1}-\gamma_{M-2}\right], \\ U_{M-2}=\left[(4I-\alpha_{M-2})\right]^{-1}\left[(\beta_{M-2}+5I)U_{M-1}+\gamma_{M-2}\right], \end{cases}$$

where $P\left[(\beta_{M-2}+5I)(4I-\alpha_{M-2})\alpha_{M-1}\right]^{-1}.$

Finally, applying second order of accuracy difference scheme (7), we obtain (9) with different matrix coefficients, where

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & d & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & d & c & d & 0 \\ 0 & 0 & 0 & 0 & \cdot & 0 & d & c & d \\ 0 & 0 & 0 & \cdot & m & n & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & r & s & t & \cdot & r & s & t \\ -1/\tau & e & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & -1/\tau & e & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ 0 & \cdot & 0 & g & f & g & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & \cdot & g & f & g \\ 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & p & q & \cdot & \cdot & 0 & 0 & 0 & \cdot & 0 & 0 & 0 \end{bmatrix}_{(2N+1)\times(2N+1)},$$

$D=B$, $E=A$ and R is $(2N+1)\times(2N+1)$ identity matrix and

Also here,

$$U_s=\begin{bmatrix} U_s^{-N} \\ \cdots \\ U_s^{0} \\ \cdots \\ U_s^{N} \end{bmatrix}_{(2N+1)\times 1}, \quad \varphi_n=\begin{bmatrix} \varphi_n^{-N} \\ \cdots \\ \varphi_n^{0} \\ \cdots \\ \varphi_n^{N} \end{bmatrix}_{(2N+1)\times 1} \quad \text{for } s=n\pm 2, n\pm 1, n.$$

Also here,

$$\begin{cases} a=\dfrac{\tau}{2h^4}, b=-\dfrac{1}{h^2}-\dfrac{2\tau}{h^4}, c=-\dfrac{1}{2h^2}, d=-\dfrac{1}{4h^2}, m=\dfrac{\tau-2}{2h^2}, \\ n=-\dfrac{\tau}{h^2}, e=\dfrac{1}{\tau}+\dfrac{2}{h^2}+\dfrac{3\tau}{h^4}, f=-\dfrac{2}{\tau^2}+\dfrac{1}{h^2}, g=\dfrac{1}{\tau^2}+\dfrac{1}{2h^2}, \\ p=-\dfrac{1}{\tau}+\dfrac{3\tau+2}{h^2}, q=\dfrac{1}{\tau}+\dfrac{2\tau}{h^2}, r=-\dfrac{1}{8\tau}, s=\dfrac{1}{2\tau}, t=-\dfrac{1}{4}-\dfrac{3}{8\tau}, \end{cases}$$

and

$$\varphi_n^k = \begin{cases} \frac{5}{4e}\sin(\pi x_n), k=-N, \\ g\left(t_k-\frac{\tau}{2},x_n\right)-\frac{\tau}{2h^2}\left(g\left(t_k-\frac{\tau}{2},x_{n+1}\right)+g\left(t_k-\frac{\tau}{2},x_n\right)+g\left(t_k-\frac{\tau}{2},x_{n-1}\right)\right), \\ -N+1\le k\le 0, \\ f(t_k,x_n), 1\le k\le N-1, \\ 0, k=N. \end{cases}$$

So, we have the fourth order difference equation with respect to n with matrix coefficients. To solve this difference equation, we have applied same procedure of modified Gauss elimination method for difference equation with respect to n with matrix coefficients. Hence, we use formulas (10) and (11) for finding of u_n^k.

Now, the result of the numerical analysis is given. For their comparison errors computed by

$$E_M^N = \max_{1\le k\le N-1}\left(\sum_{n=1}^{M-1}\left|u(t_k,x_n)-u_n^k\right|^2 h\right)^{1/2},$$

of numerical solutions are recorded for different values of N and M, where $u(t_k,x_n)$ represents the exact solution and u_n^k represents the numerical solution at (t_k,x_n). The result are shown in the Table 1 for $N=M=10,20,30,40,50$ and 60, respectively.

TABLE 1: Comparison or errors for the approximate solution of difference schemes

Method	N=10 M=20	N=20 M=40	N=40 M=80
$DS(5)$	0.1555	0.0948	0.0549
$DS(6)$	0.0846	0.0157	0.0041
$DS(7)$	0.0908	0.0182	0.0042

In conclusion, the second order of accuracy difference schemes are more accurate comparing with the first order of accuracy difference scheme.

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